

CONSTRAINT ELIMINATION IN DYNAMICAL SYSTEMS

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THEME

Large Space Structures (LSS) and other dynamical systems of current interest are often extremely complex assemblies of rigid and flexible bodies subjected to kinematical constraints. This paper presents a formulation of the governing equations of constrained multibody systems via the application of singular value decomposition (SVD). The resulting equations of motion are shown to be of minimum dimension.

The motivation for this work was the development of a generic computer program for simulating space structures and similar electromechanical systems amenable to mathematical representation as a set of flexible bodies interconnected in a topological configuration. This representation may include closed loops of bodies, prescribed motion, or other constraints that may qualify as simple monoholonomic. The equations of motion appropriate for a set of flexible bodies in an open loop configuration appear in Refs. 1, 2. A computer program (TREETOPS) developed to simulate the dynamic response of flexible structures in a topological tree configuration is described in Ref. 3. The SVD technique of the present paper is being incorporated in an extension of the TREETOPS program that permits application to constrained systems. This extension permits direct use of the dynamical equations for the less constrained system in Refs. 1, 2, with augmentation by kinematical constraint equations and reduction of dimension by SVD.

Basically, there are two conceptual approaches to solving the equations of motion of such systems. (1) One can introduce unknown forces and torques at the interfaces between constrained bodies (often accomplishing this symbolically with Lagrange multipliers), and then solve the dynamical equations simultaneously with the constraint equations to determine the constraint forces and torques as well as the kinematical variables, Ref. 4. (2) Alternatively, one can use the constraint equations to reduce the dimension of the system of dynamical equations to be solved by partitioning generalized coordinates, Refs. 5, 6. Techniques presented in Refs. 4, 5, 6 may encounter numerical singularities. Also, systems undergoing large motion may present problems of inconsistency in the constraints such as three-dimensional loops during the system motion becoming two dimensional or one-dimensional loops. In what follows, the SVD method will be shown to avoid mathematical singularities.

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CONTENTS

Singular Value Decomposition: Orthogonal decomposition of an $m \times n$ matrix L by singular value decomposition is closely related to the eigenvalue-eigenvector decomposition of the symmetric positive semidefinite matrices $L^T L$ and LL^T . Let $r \leq m$ be the rank of L . Then there are orthogonal matrices U and V of order $m \times m$ and $n \times n$ respectively such that

$$\begin{aligned} U^T L V &= \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \\ &= S \end{aligned} \quad (1)$$

where $\Sigma = \text{diag} (\lambda_1, \lambda_2, \dots, \lambda_r)$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$.

The diagonal elements of the decomposition are called the singular values of the matrix L . The singular values are unique, although U and V are not.

It is easy to verify that

$$V^T L^T L V = \text{diag} (\Sigma^2, 0) \quad (2)$$

Thus $(\lambda_1^2, \dots, \lambda_r^2)$ must be the nonzero eigenvalues of $L^T L$ arranged in the descending order and the requirement that λ_i be nonnegative completely determines the λ_i . The eigenvectors of $L^T L$ are the columns of V . If $L^T L$ has a multiple eigenvalue $\lambda^2 > 0$, the corresponding columns of V may be chosen as an orthonormal basis for the space spanned by the eigenvectors corresponding to λ^2 .

From eq. (1)

$$L = U S V^T \quad (3)$$

Now with proper partitioning of U and V eq. (3) can be expressed as

$$L = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma V_1^T \quad (4)$$

From the above one obtains

$$U_1 = L V_1 \Sigma^{-1} \quad (5)$$

Thus once V_1 is chosen U_1 is obtained by eq. (5). The matrices U_2 and V_2 may be any matrices with orthonormal columns spanning the null spaces of L^T and L , respectively. It is worthwhile to mention that the null space of L is the space of all vectors x such that

$$Lx = 0 \quad (6)$$

With the orthogonal decomposition given by eq. (3), an $n \times m$ matrix L^+ , called the pseudoinverse of L , is defined by

$$L^+ = V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T \quad (7)$$

L^+ is uniquely defined by L ; it does not depend on the particular orthogonal decomposition of L .

Application of SVD to Dynamical System with Constraints: Let $q = q_1, \dots, q_n$ comprise a set of generalized coordinates that fully defines the configuration of the dynamical system. The equations of motion of the system can be written as

$$M\ddot{q} = F(\dot{q}, q, t) \quad (8a)$$

where the elements of $n \times n$ matrix M are functions of q 's and the inertia properties of the system; the elements of $n \times 1$ column vector F are functions of q 's, their time derivatives \dot{q} 's and applied forces (moments) on the systems. If the generalized coordinates are related by constraint equations then they are not independent and the right hand side of eq. (8a) will also include the non-working forces of constraints. Let the unknown constraint forces be denoted F^C . Now for the general case of constrained dynamical system, eq. (8a) takes the following form

$$M\ddot{q} = F + F^C \quad (8)$$

Suppose however that the constraint equations can be written as

$$A\dot{q} = B \quad (9)$$

where A is of dimension $m \times n$ ($m < n$) and B is an $m \times 1$ column vector.

Holonomic constraint equations can always be placed in the form of eq. (9) and nonholonomic constraints in the class called Pfaffian or simple have this structure also.

If the rank of matrix A is $r \leq m$ then r of the kinematical variables in q are related by eq. (9) and there are only $n-r$ independent generalized coordinates. In other words the dynamical system possesses $n-r$ degrees of freedom.

The SVD of the $m \times n$ matrix A provides

$$A = USV^T \quad (10)$$

The orthogonal matrices U and V (of dimension $m \times m$ and $n \times n$, respectively) are partitioned as

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \quad (11)$$

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \quad (12)$$

where U_1 and V_1 are respectively $m \times r$ and $n \times r$ matrices; U_2 and V_2 are respectively $m \times (m-r)$ and $n \times (n-r)$ matrices. Note that r is the rank of A .

Because $AV_2 = 0$, eq. (9) is satisfied by

$$\dot{q} = A^+B + V_2\dot{z} \quad (13)$$

for any vector \dot{z} , A^+ is the pseudoinverse of A . We shall refer to z as the reduced set of $(n-r)$ coordinates.

Differentiation of eq. (9) with respect to time yields

$$A\ddot{q} = -\dot{A}\dot{q} + \dot{B} \quad (14)$$

or, $A\ddot{q} = B'$

Following eq. (13) express \ddot{q} in terms of \ddot{z} as

$$\ddot{q} = A^+B' + V_2\ddot{z} \quad (15)$$

Note from eq. (13) or eq. (15) that V_2 maps the n kinematic variables \dot{q} (or \ddot{q}) to $n-r$ variables \dot{z} (or \ddot{z}). Thus a consistent set of equations of motion in \ddot{z} is given as

$$V_2^T M V_2 \ddot{z} = V_2^T F + V_2^T F^C - V_2^T M A^+ B' \quad (16)$$

The coefficient of \ddot{z} is a symmetric positive definite matrix with the characteristic of an "inertia matrix" for the reduced set of coordinates z .

With the Lagrange multiplier method, F^C is established via (see Ref. 4)

$$F^C = A^T \alpha \quad (17)$$

where α is the column vector of Lagrange multipliers.

Premultiply eq. (17) by V_2^T to obtain the following

$$\begin{aligned} V_2^T F^C &= V_2^T A^T \alpha \\ &= (AV_2)^T \alpha \\ &= 0 \end{aligned} \quad (18)$$

Thus it is seen that the nonworking constraint forces make no contribution to the equations of motion (eq. (16)) and need not be recorded.

Employing the transformations given by eqs. (13) and (15), the minimum dimension governing differential equations of motion are given by

$$V_2^T M V_2 \ddot{z} = V_2^T F - V_2^T M A^+ B' \quad (19)$$

and

$$\dot{q} = A^+ B + V_2 \dot{z} \quad (20)$$

This method eliminates the forces of constraints which when included serve not only to enlarge the dimension of the dynamical system but also quite often introduce computational problems.

ACKNOWLEDGMENT

An excellent treatment of the computational efficiency of the SVD is given in Ref. 7.

REFERENCES

1. Singh, R. P. and Likins, P. W., "Manipulator Interactive Design with Interconnected Flexible Elements", Proceedings of the 1983 Automatic Control Conference, San Francisco, CA., June 1983, pp. 505-512.
2. Singh, R. P., VanderVoort, R. J. and Likins, P. W., "Dynamics of Flexible Bodies in Tree Topology - A Computer Oriented Approach", Proceedings of the 25th SDM and AIAA Dynamics Specialist Conference, Palm Springs, CA., May 1984, Part 2, pp. 327-337.
3. VanderVoort, R. J. and Singh, R. P., "A Pointing Control System Simulation and Generic Model Generator for Multi-Body Flexible Structures" Proceedings of the Ninth DARPA Strategic and Space Symposium, Monterey, CA., October 1983.
4. Bodley, C., Devers, A., Park A. and Frisch, H., "A Digital Computer Program for Dynamic Interaction Simulation of Controls and Structures (DISCOS)", NASA Technical Paper 1219, May 1978.
5. Kane, T. R., and Wang, C. F., "On the Derivation of Equations of Motion", J. SIAM, Vol. 13, 1965, pp. 487 - 492.
6. Wehage, R. A. and Haug, E. J., "Generalized Coordinate Partitioning for Dimension Reduction in Analysis of Constrained Dynamic Systems", ASME Journal of Mechanical Design, Vol. 104, January 1982, pp. 247-255.
7. Juang, Jer-Nan and Pinson, Larry D., "Application of Singular Value Decomposition to Structural Dynamics Systems with Constraints," AIAA Paper No. 85-0687-CP, April 1985.